A New Algorithm for Discrete Area of Convex Polygons with Rational Vertices

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Abstract. A new algorithm is presented, which computes the number of lattice points lying inside a convex plane polygon from the sequence of the rational coordinates of its vertices. It reduces the general case in a natural way to a fondamental one, namely a triangle with vertices of coordinates $\{(0;0),(n;0),(n;n\frac{a}{b})\}$, where n, a and b are positive natural integers. Then it evaluates the discrete area of such a triangle using the Klein polyhedron of slope $\frac{a}{b}$ and the Ostrowski representation of n with the numeration scale of denominators of the convergents of the continued fraction expansion of $\frac{a}{b}$.

Keywords: counting, lattice points, rational polygon, Klein polyhedra.

Introduction

A wide variety of topics involve the problem of counting the number of lattice points (i.e. points with integer coordinates) inside a convex bounded polygon: number theory, toric Hilbert functions, Kostant's partition function in representation theory, cryptography, integer programming,... (for details, see [**?**]).

In 1993 A. Barvinok [**?**] found an algorithm to count integer points inside polyhedra for each fixed dimension in polynomial time on the size of the input (the size is given by the binary encoding of the data).

The key ideas are using rational functions and the unimodular signed decomposition of polyhedra. Given a convex polyhedron P in a d dimensional space, it computes the multivariate generating function

$$f(P;x) = \sum_{\alpha \in P \cap \mathbb{Z}^d} x^\alpha$$

where $x^\alpha = x_1^{\alpha_1}...x_d^{\alpha_d}$.

The number of lattice point of a polytope will be the value of its generating function at $(1,...,1)$. It turns out that it is easy to compute the generating function of a unimodular cone (i.e. a cone with the vectors of a basis of the lattice as generating vectors). But now a theorem of M. Brion [**?**] says that to compute the rational function representation of $f(P;z)$, it is enough to do it for tangent cones at each vertex of P. Let P be a convex polytope and let $V(P)$ be the vertex set of P. Let Kv be the tangent cone at $v \in V(P)$, which is the (possibly translated) cone defined by the facets touching vertex v. Then the following formula holds: $f(P;z) = \sum_{v \in V(P)} f(K_v; z)$. Lastly Barvinok's algorithm provides a signed decomposition of the K_v as disjoint unions of unimodular cones.

In [**?**], M. Beck and S. Robins give explicit polytime-computable formulae for the number of lattice points in two-dimensional rational polytope. This method uses Dedekind-Rademacher sums.

Building on ideas of J.-P. Réveillès in [**?**], we propose here a new algorithm working in two-dimensional space. Its input is the sequence of rational coordinates of the vertices of the convex polygon. It reduces the general case in a natural way to a fondamental one, namely a triangle with vertices of coordinates $\{(0;0),(n;0),(n;n\frac{a}{b})\}$, where n, a and b are positive natural integers. Dealing with this fundamental case involves classical properties of the geometry of numbers, such as geometrical interpretation of the continued fractions, Klein's theorem, Ostrowski's reprentation of natural numbers. We need also the classical Pick's formula, which is in some sense generalized. The first part is devoted to mathematical tools. In the second part, the algorithm is described.

1 Mathematical tools

We first need classical notions from continued fractions theory (see for example the classical textbook [**?**]).

Let a and b be coprime positive integers. Then we denote α_i and r_i the i^{th} term of the sequence of quotients and remainders of the classical Euclidean algorithm:

$$r_0 = a, \qquad r_1 = b, \qquad \text{and while } r_{i+1} \neq 0, \begin{cases} r_i = \alpha_i r_{i+1} + r_{i+2} \\ 0 \leq r_{i+2} < r_{i+1} \end{cases}$$

The length l of the algorithm is the number of steps $l = Sup\{i; r_i \neq 0\}$. Let us define the sequence of continued fraction expansion of $\dfrac{a}{b}$ in the following way:

$$\begin{array}{cc} \alpha_k p_{k-1} + p_{k-2} = p_k & \qquad \alpha_k q_{k-1} + q_{k-2} = q_k \\ p_{-2} = 0, p_{-1} = 1 & \qquad q_{-2} = 1, q_{-1} = 0 \end{array}$$

$\left(\frac{p_i}{q_i}\right)_{0 \leq i \leq \lambda}$ is the sequence of the convergents of $\dfrac{a}{b}$. We have $b = q_{l-1}$ and $a = p_{l-1}$ and $\frac{p_{2k}}{q_{2k}} < \frac{a}{b} < \frac{p_{2k+1}}{q_{2k+1}}$ for $2k+1 \leq l-2$.

The number of lattice points lying in a polygon is called its discrete area. The well known Pick's formula ([**?**]) is the following:

Lemma 1. *The discrete area of a polygon with lattice points as vertices is equal to its continuous area plus half of the number of the lattice points lying on its boundary minus 1.*

Notice that vertices must be lattice points. Our problem is to generalize this formula to polygons with rational vertices. One of the ideas we largely use is to substitute to such a lattice polygon a larger or smaller polygon (for the order of inclusion) having the same discrete area and lattice points as vertices. The two following lemmas provide opportunities to do so.

Lemma 2. *There is no lattice points such that* $\gamma < \alpha x+\beta y < gcd(\alpha,\beta)\lceil \frac{\gamma}{gcd(\alpha,\beta)} \rceil$ *where* α, β, γ *are integers and* $gcd(\alpha, \beta)$ *is the greatest common divisor of* α *and* β.

Proof. Let r be such that $0 \leq r < gcd(\alpha x, \beta)$ and $\gamma = gcd(\alpha,\beta)\lceil \frac{\gamma}{gcd(\alpha,\beta)} \rceil - r$. Then the given condition implies $r > gcd(\alpha,\beta)\lceil \frac{\gamma}{gcd(\alpha,\beta)} \rceil - (\alpha x + \beta y) > 0$, which is impossible since the middle term is a multiple of $gcd(\alpha, \beta)$.

The following is an old theorm of Klein (see [**?**])

Lemma 3. *There is no lattice points in the polygon of vertices of coordinates* $(q_i; \frac{a}{b}q_i)$, $(q_{i+2}; \frac{a}{b}q_{i+2})$, $(q_{i+2}; p_{i+2})$ *and* $(q_i; p_i)$ *but the points of coordinates* $(q_i + tq_{i+1}; p_i + tp_{i+1})$ *where* $0 \leq t \leq \alpha_{i+2}$.

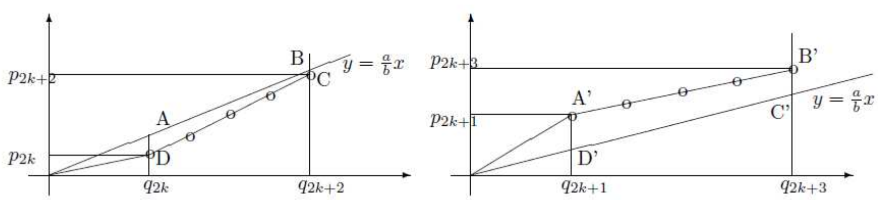


Fig. 1. Left: $i = 2k$ is even Right: $i = 2k + 1$ is odd

Let us denote by $A(n; a, b)$ the discrete area of the triangle $\{(x; y) \in \mathbb{N}^2; (0 \leq x \leq n) \wedge \left(0 \leq y \leq \frac{a}{b}x\right)\}$. Formulas will differ, depending on the parity of the indices of the convergents of $\dfrac{a}{b}$. Let us denote by $i\%2$ the remainder of the integer i in the euclidean division by 2.

Lemma 4. *Let us denote by* $T(i, t; a, b) = A(q_i+tq_{i+1}-1; a, b)-A(q_i-1; a, b)$ *the discrete area of* $\left\{(x; y) \in \mathbb{Z}^2; (q_i \leq x \leq q_i + tq_{i+1} - 1) \wedge \left(0 \leq y \leq \dfrac{a}{b}x\right)\right\}$. *Then*

$$T(i, t; a, b) = \frac{t}{2}\left((2p_i + tp_{i+1})q_{i+1} + q_{i+1} - p_{i+1} + 1 - 2(i\%2)\right)$$

for $0 \leq t \leq \alpha_{i+2}$ *and* $i + 2 \leq l$.

Proof. From Klein's theorem, there is no lattice points in the parallelogram of vertices of coordinates $(q_{2k}; \frac{a}{b}q_{2k})$, $(q_{2k+2}; \frac{a}{b}q_{2k+2})$, $(q_{2k+2}; p_{2k+2})$ and $(q_{2k}; p_{2k})$ but the points of coordinates $(q_{2k} + tq_{2k+1}; p_{2k} + tp_{2k+1})$ where $0 \leq t \leq \alpha_{2k+2}$ (see 1, Left). Applying now Pick's formula, $T(2k, t; a, b)$ turns out to be equal to

$$\frac{t}{2}\left((2p_{2k} + tp_{2k+1})q_{2k+1} + q_{2k+1} - p_{2k+1} + 1\right)$$

which is the announced result for even i.

A similar proof (see figure 1, Right) shows that $T(2k+1, t; a, b)$ is equal to

$$\frac{t}{2}\left((2p_{2k+1} + tp_{2k+2})q_{2k+2} + q_{2k+2} - p_{2k+2} - 1\right)$$

for $0 \leq t \leq \alpha_{2k+3}$ and $2k+3 \leq b$.

Next lemma is devoted to $\left\{(x; y) \in \mathbb{Z}^2; (q_i + tq_{i+1} \leq x \leq n) \wedge \left(0 \leq y \leq \frac{a}{b}x\right)\right\}$.

Lemma 5. *Suppose* $q_i + tq_{i+1} + 1 \leq n \leq q_i + (t+1)q_{i+1} - 1$ *for* $0 \leq t \leq \alpha_{i+2}$ *; then* $A(n; a, b) - A(q_i + tq_{i+1} - 1; a, b)$ *is equal to*

$$A(n - q_i - tq_{i+1}; p_{i+1}, q_{i+1}) + (n - q_i - tq_{i+1} + 1)(p_i + tp_{i+1}) - (i\%2)$$

Proof. (1) let us first consider the even case $i = 2k$; as a consequence of Klein's theorem, the set of the lattice points satisfying $(q_{2k} + tq_{2k+1} \leq x \leq n) \wedge \left(0 \leq y \leq \frac{a}{b}x\right)$ is the same than the set of the lattice points satisfying $(q_{2k} + tq_{2k+1} \leq x \leq n) \wedge \left(0 \leq y \leq p_{2k} + tp_{2k+1} + \frac{p_{2k+1}}{q_{2k+1}}(x - q_{2k} - tq_{2k+1})\right)$ which is the disjoint union of the rectangle defined by $(q_{2k} + tq_{2k+1} \leq x \leq n) \wedge (0 \leq y \leq p_{2k} + tp_{2k+1} - 1)$ and the triangle defined by $(q_{2k} + tq_{2k+1} \leq x \leq n) \wedge \left(p_{2k} + tp_{2k+1} \leq y \leq p_{2k} + tp_{2k+1} + \frac{p_{2k+1}}{q_{2k+1}}(x - q_{2k} - tq_{2k+1})\right)$. But the discrete area of first one is $(n - q_{2k} - tq_{2k+1} + 1) \times (p_{2k} + tp_{2k+1})$. And the second one have the same discrete area than the triangle $(0 \leq x \leq n - q_{2k} - tq_{2k+1}) \wedge \left(0 \leq y \leq \frac{p_{2k+1}}{q_{2k+1}}x\right)$.

(2) in the odd case $i = 2k+1$, the proof is similar, just taking care to the point of coordinates $(q_{2k+1} + tq_{2k+2}; p_{2k+1} + tp_{2k+2})$.

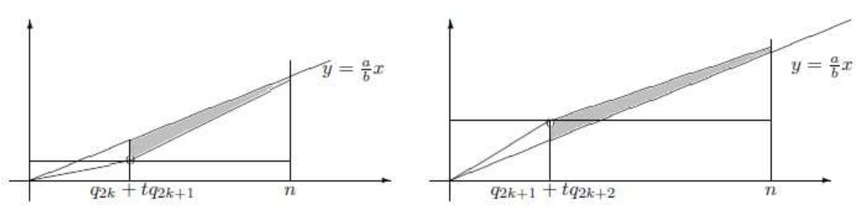


Fig. 2. Left $i = 2k$ Right $i = 2k+1$

In order to initialize an induction, we need values of $A(n; a, b)$ for small n.

Lemma 6. *For* $0 \leq n \leq q_1$ *we have* $A(n; a, b) = (n+1) + p_0\dfrac{n(n+1)}{2}$.

Proof. $A(n; a, b) = 1 + (p_0 + 1) + ... + ((n-1)p_0 + 1)$

Let us introduce now the usual Ostrowski representations of integers [?], [?], [?]. With the previous notations, for $1 \leq n \leq b-1$, there exist a finite sequence of non negative integers $(n_i)_{0\leq i\leq\lambda-1}$ such that $n = \sum_{i=0}^{i=\lambda-1} n_i q_i$ and $0 \leq n_0 \leq \alpha_1$ and $0 \leq n_i \leq \alpha_i$ and $n_i = \alpha_{i+1}$ implies $n_{i-1} = 0$ and $n_{\lambda-1} \neq 0$. For our purpose, we need a slightly different form:

Lemma 7. *For $1 \leq n \leq b-1$, there exist finite sequences of integers $(\nu_i)_{0\leq i\leq\lambda-1}$ and $(\phi_i)_{0\leq i\leq\lambda-1}$, the second one being increasing, such that*

$$n = \sum_{i=0}^{i=\lambda-1} \nu_i q_{\phi_i} \tag{1}$$

and $0 < \nu_i \leq \alpha_i$ and $\phi_{i-1} \leq \phi_i - 2$ implies $\nu_i = 1$ and $\nu_{\lambda-1} \neq 0$.

Proof. From the canonical representation, do inductively from the digit of maximal weight $n_{\lambda-1}$:
if $n_i = 0$ and $n_{i+1} \geq 2$, replace $n_{i+1}q_{i+1} + 0q_i + n_{i-1}q_{i-1}$ by $(n_{i+1}-1)q_{i+1} + \alpha_{i+1}q_i + (n_{i-1}+1)q_{i-1}$.
if $n_{i-1} \leq \alpha_i - 1$, stop;
if $n_{i-1} = \alpha_i$, then n_{i-2} is equal to 0; hence it is possible to iterate.
In fine, just consider an increasing enumeration ϕ of the index of non zero digits n_i.

We are able now to give the main theorem.

Theorem 1. *Suppose $1 \leq n \leq b-1$.*
If $\phi_{\lambda-2} = \phi_{\lambda-1} - 1$, then $A(n;a,b)$ is equal to

$$A(q_{\phi_{\lambda-2}} + n_{\lambda-1}q_{\phi_{\lambda-1}} - 1; a, b) + (n - q_{\phi_{\lambda-2}} - \nu_{\lambda-1}q_{\phi_{\lambda-1}} + 1)(p_{\phi_{\lambda-2}} + \nu_{\lambda-1}p_{\phi_{\lambda-1}})$$
$$-(\phi_{\lambda-1}\%2) + A(n - q_{\phi_{\lambda-2}} - \nu_{\lambda-1}q_{\phi_{\lambda-1}}; p_{\phi_{\lambda-1}}, q_{\phi_{\lambda-1}})$$

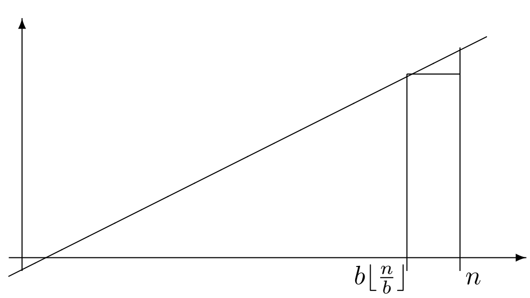
If $\phi_{\lambda-2} \leq \phi_{\lambda-1} - 2$, then $A(n;a,b)$ is equal to

$$A(q_{\phi_{\lambda-1}} - 1; a, b) + (n - q_{\phi_{\lambda-1}} + 1)p_{\phi_{\lambda-1}} - (\phi_{\lambda-1}\%2) + A(n - q_{\phi_{\lambda-1}}; p_{\phi_{\lambda-1}}, q_{\phi_{\lambda-1}})$$

Lastly, we generalize easily to any positive integers n

Lemma 8. *Suppose $b \leq n$, then $A(n;a,b)$ is equal to*

$$A(n - b\lfloor\frac{n}{b}\rfloor; a, b) - \lfloor\frac{n}{b}\rfloor^2 \frac{ab}{2} + \frac{1}{2}\lfloor\frac{n}{b}\rfloor(b - a + 1 + 2an)$$



Proof. $A(n;a,b)$ is equal to $A(n-b\lfloor\frac{n}{b}\rfloor;a,b)$ plus the discrete area of the rectangle $(b\lfloor\frac{n}{b}\rfloor \leq x < n) \wedge (0 \leq y < a\lfloor\frac{n}{b}\rfloor)$ which is

$$\frac{1}{2}b\lfloor\frac{n}{b}\rfloor a\lfloor\frac{n}{b}\rfloor + \frac{1}{2}(\lfloor\frac{n}{b}\rfloor + a\lfloor\frac{n}{b}\rfloor + b\lfloor\frac{n}{b}\rfloor) + 1 - (1 + b\lfloor\frac{n}{b}\rfloor) + (n - b\lfloor\frac{n}{b}\rfloor)a\lfloor\frac{n}{b}\rfloor$$

which is equal to $-\frac{1}{2}b\lfloor\frac{n}{b}\rfloor a\lfloor\frac{n}{b}\rfloor + \frac{1}{2}(\lfloor\frac{n}{b}\rfloor + a\lfloor\frac{n}{b}\rfloor - b\lfloor\frac{n}{b}\rfloor) + na\lfloor\frac{n}{b}\rfloor$. QED

2 Algorithm

In this section, we first present the algorithm in the case of a triangle with vertices of coordinates $\{(0;0),(n;0),(n;n\frac{a}{b})\}$, where n, a and b are positive natural integers. This algorithm follows from the mathematical tools. The second subsection reduces the general case to this fundamental case. The third deals with complexity.

2.1 Fundamental triangle

The function `T(i,t)` computinges $T(i,t;a,b)$. The function `AA(i,t)` computes $A(q_i + tq_{i+1} - 1;a,b)$. The complete algorithm corresponds to the function `algo(n)`. At each step, it realizes the induction described in Theorem 8. However it does not compute all the digits of the Ostrowski representation of n before the induction: each digit is computed at the corresponding step of the induction.

```
Input: a, b, n
Using Extended Euclidean algorithm, compute
   l         ♯   the length of the continued fraction
             ♯   expansion of a/b
   listea    ♯   the list of the coefficients of the continued
             ♯   fraction expansion of a/b
   listep    ♯   the list of the numerators of the
             ♯   convergents of a/b
   listeq    ♯   the list of the denominators of the
             ♯   convergents of a/b
def T(i,t):
  T=t*(listeq[i+1]*(2*listep[i]+t*listep[i+1])
                    -listep[i+1]+listeq[i+1]+1-2*(i%2)))//2
  return (T)

def AA(i,t):
  Sum=0
  Sum=1+(i%2)*((listep[0]*(listeq[1]*(listeq[1]-1)))//2+listeq[1]-1)
  j=i%2
  while j < i-1:
   Sum = Sum + T(j,listea[j+2])
```

```
    j=j+2
  Sum = Sum + T(i,t)
   return Sum
def algo(n):
  Sum=0
  while n>= listeq[1]:
    j=0
    while listeq[j]<=n:
      j=j+1
    c=n//listeq[j-1]
    d=n-c * listeq[j-1]
    k=0
    while listeq[k]<=d:
      k=k+1
    if k==j-1:
      s=AA(j-2,c)+(n-listeq[j-2]-c*listeq[j-1]+1)*
                    (listep[j-2]+c*listep[j-1])-((j-2)%2)
      Sum = Sum + s
      n=n-listeq[j-2]-c*listeq[j-1]
    else:
      s=AA(j-2,c-1)+(n-listeq[j-2]-(c-1)*listeq[j-1]+1)
               *(listep[j-2]+(c-1)*listep[j-1])-((j-2)%2)
      Sum = Sum + s
       n=n-listeq[j-2]-(c-1)*listeq[j-1]
  Sum = Sum + n+1+listep[0]*(n*(n+1))//2
  return Sum
```

2.2 Reduction of convex polygons to fundamental triangles

This part of the algorithm is straightforward, so we just sketch the outline. Let a rational convex polygon P be defined by a sequence of its vertices $(v_i)_{0\leq i\leq i_{max}}$. Each rational vertex v_i will be defined by its coordinates, and each rational coordinates will be defined by an irreducible fraction given as a couple of coprime integers $(Nx; Dx)$, the first one beeing its numerator and the second one its denominator. The *first step* consits in providing a counterclock wise ordering of the vertices and indices of vertices of extremal abscissae and ordinates.

Hence the output will be a sequence $(x_i, y_i)_{0\leq i\leq i_{max}}$ and four indices 0, i_1, i_2 and i_3 such that for $0 \leq i \leq i_1 - 1$ we have $x_i \geq x_{i+1}$ and $y_i \geq y_{i+1}$, for $i_1 \leq i \leq i_2 - 1$ we have $x_i \leq x_{i+1}$ and $y_i \geq y_{i+1}$, and so on.

We introduce now the rectangle R defined by $(x_{i_1} \leq x \leq x_{i_3}) \wedge (y_{i_2} \leq y \leq y_0)$. The idea is to substract to the number of lattice points of R the number of lattice points outside P. The *second step* provides a decomposition of $R-P$ as a union of open triangles and rectangles and their boudaries: the rectangles will be defined for $0 \leq i \leq i_1 - 1$ by $(x_{i_1} < x < x_{i+1}) \wedge (y_{i+1} < y < y_i)$, for $i_1 \leq i \leq i_2 - 1$ defined by $(x_i < x < x_{i+1}) \wedge (y_{i_1} < y < y_{i+1})$, and so on; the triangles will be defined by vertices v_i, v_{i+1}, and w_i where w_i is of coordinates (x_{i+1}, y_i) for $0 \leq i \leq i_1 - 1$, and of coordinates (x_i, y_{i+1}) for $i_1 \leq i \leq i_2 - 1$, and so on.

Counting the lattice points in the rectangles and on their boundaries is easy. We focuse now on the triangles. Each is easily reduced to a more simple triangle with vertices of type $(0;0), (r_1;0), (0;r_2)$, where r_1 and r_2 are positive rational numbers. Now we reduce such triangles to fundamental ones. Suppose that $\alpha x + \beta y = \gamma$ is an equation of the line defined by $(r_1;0)$ and $(0;r_2)$, with α , β and γ positive integers. More precisely, $\alpha = Nx_2Dx_1$, $\beta = Nx_1Dx_2$, $\gamma = Nx_1Nx_2$. So the triangle is defined by $(0 < x) \wedge (0 < y) \wedge (\alpha x + \beta y < \gamma)$.

The easy case is whenever there is a lattice point on the straightline defined by $\alpha x + \beta y \leq \gamma$. It happens iff $gcd(\alpha, \beta)$ divides γ. Let us introduce $\alpha' = \frac{\alpha}{gcd(\alpha,\beta)}$ and $\beta' = \frac{\beta}{gcd(\alpha,\beta)}$. Let $(x_0; y_0)$ be an integer solution. One can get it using Extended Euclidean algorithm and choose it such as $r_1 \leq x_0 < r_1 + \beta$ and $y_0 \leq 0$. In this case the number of lattice points inside the open triangle $(0 < x) \wedge (0 < y) \wedge (\alpha x + \beta y < \gamma)$ is $A(x_0; \alpha', \beta') - A(x_0 - \lfloor r_1 \rfloor - 1; \alpha', \beta') - (\lfloor r_1 \rfloor + 1)y_0 - \lfloor r_2 \rfloor - \lfloor \frac{x_0\gamma}{\beta} \rfloor - \lfloor \frac{y_0\gamma}{\alpha} \rfloor - 1$

We have thus to consider the more difficult case where the gcd of α and β does not divide γ. In this case, the considered triangle has the same number of lattice points than the triangle defined by

$$(0 < x) \wedge (0 < y) \wedge \left(\alpha' x + \beta' y < \lceil \frac{\gamma}{gcd(\alpha, \beta)} \rceil \right)$$

2.3 Complexity

The algorithm presented in this paper is of restricted application, just counting lattice points in convex plane polygonals. The main step is the so-called fundamental case. Its complexity is $O(d^2 \times \tau(d))$, where d is the maximal size of the input a and b and $\tau(d)$ the time required for multiplying or dividing integers of same size.

This algorithm have been implemented in **Python** on a P.C. equipped with a Intel Cetrino at 2.53 GHz. Average time computation for input with 7 decimal digits is almost 10^{-4} s and for input with 17 decimal digits almost 5×10^{-4} s

3 Conclusion

Barvinok's algorithm is implemented in two packages: `LattE` (described in [**?**]) and `Barvinok` (described in [**?**]). Both are dealing with polytopes in arbitrary fixed dimension and not only compute the number of lattice points, but compute the Ehrhard polynomials, optimize functions on the lattice points of the polytopes, ... Moreover they are implemented with gmp integers. Nevertheless our algorithm is of theoretical interest because of its simplcity, which could leads to a $LOGSPACE$ complexity. Moreover, it is of significant efficiency in the two-dimensional case.

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